RELATIONSHIPS BETWEEN THE WIND-FIELD, THE TRANSPORT-FIELD AND THE MEAN MASS-FIELD IN A NON-HOMOGENEOUS OCEAN

by

V.B. Shtokman

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E.R. Hope

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RELATIONSHIPS BETWEEN THE WIND-FIELD, THE TRANSPORT FIELD AND THE MEAN MASS-FIELD IN A NON-HOMOGENEOUS OCEAN

V.B.SHTOKMAN

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The classical theory of the horizontal circulation in a homogeneous ocean, as constructed by V.EKMAN {1}, suffers from the defect that it considers only the frictional forces acting in horizontal planes. In consequence, once the water on the sea-bottom has acquired a motion, vertical velocity-gradients must be developed just as great as those in the surface·layer of the sea. This result does not agree with observed data, which show that the vertical velocity-gradients in a non-homogeneous ocean are rapidly damped out with increasing depth, and at the bottom are to all practical intents and purposes insignificant. This rapid attenuation of the vertical velocity-gradients is a result of intensive horizontal exchange and transfer of motions, setting up, in vertical planes parallel to the horizontal components of flow, large frictional forces which almost entirely counter-balance the tangential stresses due to the wind on the surface of the ocean. Thus the distinctive feature of an established motion in a non-homogeneous ocean is the fact, confirmed by observation, that the frictional forces at the bottom are extremely small in comparison with the stresses set up by the turbulent "side-friction". On the basis of these findings, the equations for the components of the steady-state transport * in a non-homogeneous ocean may be written in the following form:

$$\begin{aligned} A_l\left(\frac{\partial^2 S_x}{\partial x^2}+\frac{\partial^2 S_x}{\partial y^2}\right)+T_x+c\bar{\rho}S_y&=g\,\frac{\partial Q}{\partial x}\,, \\ A_l\left(\frac{\partial^2 S_y}{\partial x^2}+\frac{\partial^2 S_y}{\partial y^2}\right)+T_y-c\bar{\rho}S_x&=g\,\frac{\partial Q}{\partial y}\,, \end{aligned} \qquad (1)$$

* "Transport" in this translation corresponds to Ekman's Strommenge (quantity of flow), as defined in equations 1a. (Translator)

Here A_l represents the coefficient of horizontal turbulent friction (which is of the order of 10^8 CGS), $c = 2\omega \sin \varphi$ is the Coriolis parameter, g is the acceleration of gravity, T_x and T_y are the horizontal x- and y-components of the tangential wind-pressure at the surface of the ocean, $\bar{\rho}$ is the mean density of the water in a column extending from the surface ($z = 0$) to the bottom ($z = h$), and S_x and S_y are the components of the transport:

$$S_x = \int_0^h u\,dz, \qquad S_y = \int_0^h v\,dz, \tag{1a}$$

where u and v are the horizontal components of the velocity of flow. Again, we have

$$Q = \int_0^h dz \int_0^z \rho\,dz.$$

The equation of continuity for the transport may be written in the form:

$$\frac{\partial}{\partial x}(\bar{\rho} S_x) + \frac{\partial}{\partial y}(\bar{\rho} S_y) = 0.$$

Since the variations of $\bar{\rho}$ along x and y are very small in comparison with the corresponding variations of S_x and S_y, we may neglect the quantity $\bar{\rho}$ in the last equation, and write

$$\operatorname{div} S = \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = 0. \tag{2}$$

For the same reason we may, in differentiating equations (1) with respect to x and y, put $\bar{\rho}$ approximately equal to unity. Eliminating the quantities $\partial Q/\partial x$ and $\partial Q/\partial y$ from equations (1), we have by (2):---

$$\nabla^2 \operatorname{rot} S = -\operatorname{rot} T / A_l, \tag{3}$$

where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2, \ \operatorname{rot} T \equiv \partial T_y/\partial x - \partial T_x/\partial y, \ \operatorname{rot} S \equiv \partial S_y/\partial x - \partial S_x/\partial y.$$

If we introduce the transport function ψ, related to [x- and y-] components of the transport by the expressions:---

$$S_x = -\partial\psi/\partial y, \quad S_y = \partial\psi/\partial x, \tag{4}$$

then (3) is transformed into the equation

$$\nabla^4\psi = -\operatorname{rot} T/A_l, \tag{5}$$

analogous to the equilibrium-equations of a plate under the action of applied loads (to which the quantity rot T is analogous.) In equation (5), ∇^4 represents the biharmonic operator. The boundary conditions for the transport on the contour Γ of the ocean shore are analogous to those for a plate held by pinching it edge-on:---

$$(\psi)_\Gamma = 0, \quad \left(\frac{\partial\psi}{\partial n}\right)_\Gamma = 0, \tag{6}$$

where n is the normal to the contour. Conditions (6) are derived by considering that the tangential and normal components of the transport at the shore-line must reduce to zero. Let us now find the equation relating the mass-field to the wind-field. Finding S_x from the first equation of (1) and substituting it in the second equation, we solve the latter for S_y:

$$cS_y = g\frac{\partial Q}{\partial x} - T_x + \frac{A_l g}{c}\frac{\partial}{\partial y}\nabla^2 Q - \frac{A_l}{c}\nabla^2 T_y - \frac{A_l^2}{c}\nabla^4 S_y. \tag{7}$$

Similarly:

$$cS_x = -g\frac{\partial Q}{\partial y} + T_y + \frac{A_l g}{c}\frac{\partial}{\partial x}\nabla^2 Q - \frac{A_l}{c}\nabla^2 T_x - \frac{A_l^2}{c}\nabla^4 S_x. \tag{8}$$

Differentiating equation (7) with respect to y, equation (8) with respect to x, and combining the results, we get:

$$c\operatorname{div} S = \operatorname{rot} T + \frac{A_l g}{c}\nabla^4 Q - \frac{A_l}{c}\nabla^2 \operatorname{div} T - \frac{A_l^2}{c}\nabla^4 \operatorname{div} S$$

or, by condition (2):---

$$\nabla^4 Q = \frac{1}{g}\left(-\frac{c \operatorname{rot} T}{A_l} + \nabla^2 \operatorname{div} T\right). \tag{9}$$

Thus the distribution of masses, represented by Q, depends not only on the Coriolis parameter and rot T, but also on the quantity $\nabla^2 \operatorname{div} T = \operatorname{div} \operatorname{grad} \operatorname{div} T$. The latter is usually very small in comparison with the first term of (9). However, it cannot be neglected in regions of marked convergence or divergence of air-currents (for instance the center of a cyclone or anti-cyclone).

If to describe the mass-field we [choose to] use the quantity P, related to the dynamic height by the expression

$$P = \int_0^h D dz = \bar{D} h,$$

where the [geopotential] height $\bar{D}$ is measured in dynamic centimeters, then the last equation will be written in the form:

$$\nabla^4 \bar{D} = \frac{1}{10^3 h}\left(-\frac{c \operatorname{rot} T}{A_l} + \nabla^2 \operatorname{div} T\right). \tag{10}$$

The boundary conditions to which Q or $\bar{D}$ are subject will be found from equations (7) and (8). If at a given point we take x as the direction tangential to the shore-line and y as the normal thereto, and if we bear in mind that at the shore-line $S_x = S_y = 0$, then we obtain the following conditions for Q:

$$g\frac{\partial Q}{\partial x} - T_x + \frac{A_l g}{c}\nabla^2 \frac{\partial Q}{\partial y} - \frac{A_l}{c}\nabla^2 T_y - \frac{A_l^2}{c}\nabla^4 S_y = 0, \tag{11}$$

$$-g\frac{\partial Q}{\partial y} + T_y + \frac{A_l g}{c}\nabla^2 \frac{\partial Q}{\partial x} - \frac{A_l}{c}\nabla^2 T_x - \frac{A_l^2}{c}\nabla^4 S_x = 0. \tag{12}$$

Recalling (4), we may write:

$$\nabla^4 S_y = \frac{\partial}{\partial x}\nabla^4 \psi, \quad \nabla^4 S_x = -\frac{\partial}{\partial y}\nabla^4 \psi.$$

Replacing $\nabla^4\psi$ by its expression in terms of rot T as given by (5), we obtain

$$\frac{A_l^2}{c}\nabla^4 S_y = -\frac{A_l}{c}\frac{\partial}{\partial x}\,\mathrm{rot}\,T; \quad \frac{A_l^2}{c}\nabla^4 S_x = \frac{A_l}{c}\frac{\partial}{\partial y}\,\mathrm{rot}\,T. \tag{13}$$

Substituting (13) in (11) and (12), we obtain equivalent boundary conditions in the following form:---

$$\frac{\partial Q}{\partial x} = \frac{T_x}{g} - \frac{A_l}{c}\nabla^2\frac{\partial Q}{\partial y} + \frac{A_l}{cg}\nabla^2 T_y - \frac{A_l}{c}\frac{\partial}{\partial x}\,\mathrm{rot}\,T, \tag{14}$$

$$\frac{\partial Q}{\partial y} = \frac{T_y}{g} + \frac{A_l}{c}\nabla^2\frac{\partial Q}{\partial x} - \frac{A_l}{cg}\nabla^2 T_x - \frac{A_l}{cg}\frac{\partial}{\partial y}\,\mathrm{rot}\,T. \tag{15}$$

Finally, let us substitute expression (15) for $\frac{\partial Q}{\partial y}$ under the ∇^2 sign in (14):

$$\frac{\partial Q}{\partial x} = \frac{T_x}{g} - \frac{A_l}{c}\nabla^2\left(\frac{T_y}{g} + \frac{A_l}{c}\nabla^2\frac{\partial Q}{\partial x} - \frac{A_l}{cg}\nabla^2 T_x - \frac{A_l}{cg}\frac{\partial}{\partial y}\,\mathrm{rot}\,T\right) +$$
$$+ \frac{A_l}{cg}\nabla^2 T_y - \frac{A_l}{cg}\frac{\partial}{\partial x}\,\mathrm{rot}\,T.$$

Expanding the operators and reducing similar terms, we see that all terms except the first in the right-hand side of this equation cancel each other. Thus we obtain a very simple boundary condition on the shore-line contour:

$$\partial Q/\partial x = T_x/g \tag{16}$$

and in the same way, a second condition:---

$$\partial Q/\partial y = T_y/g. \tag{17}$$

If we use $\bar{D}$, the boundary conditions will obviously be written in the form:

$$\frac{\partial \bar{D}}{\partial x} = \frac{T_x}{10^3 h}; \quad \frac{\partial \bar{D}}{\partial y} = \frac{T_y}{10^3 h}. \tag{18}$$

Let us now establish the relation between the field of masses and the transport. Differentiating the first equation of (1) with respect to x, the second with respect to y, and combining the results ($\bar{\rho} = 1$), we obtain

$$g\nabla^2 Q = c \operatorname{rot} S + \operatorname{div} T + A_l \nabla^2 \operatorname{div} S$$

or, by condition (2):---

$$\nabla^2 Q = \frac{1}{g}\ (c \operatorname{rot} S + \operatorname{div} T). \tag{19}$$

From (19) it follows that the adaptation of the mass-field to the transport field takes place independently of the size of the coefficient of horizontal turbulent exchange A_l. At the equator, where $c = 0$, the adaptation of masses is regulated exclusively by the quantity div T. We note that formulae (7) and (8) may be used for calculating the transport components from observed mass-field and wind-field data, if we substitute expression (13) in (7) and (8). In this way we obtain the formulae:---

$$S_y = \frac{g}{c}\ \frac{\partial Q}{\partial x} - \frac{T_x}{c} + \frac{A_l g}{c^2}\ \frac{\partial}{\partial y} \nabla^2 Q - \frac{A_l}{c^2}\ \nabla^2 T_y + \frac{A_l}{c^2}\ \frac{\partial}{\partial x} \operatorname{rot} T, \tag{20}$$

$$S_x = -\ \frac{g}{c}\frac{\partial Q}{\partial y} + \frac{T_y}{c} + \frac{A_l g}{c^2}\ \frac{\partial}{\partial x} \nabla^2 Q - \frac{A_l}{c^2}\ \nabla^2 T_x - \frac{A_l}{c^2}\ \frac{\partial}{\partial y} \operatorname{rot} T, \tag{21}$$

replacing the well-known formulae of Ekman {2} and Jakhelln {3}. An estimate of the order of magnitude of the terms in (20) and (21) will show that the first three terms on the right-hand side are often identical in amount.

Thus if we take "side-friction" and wind into account, it may introduce substantial corrections in Ekman's formulae. When there is no wind, or when its effect may be neglected, expressions (20) and (21) are a good deal simplified.

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Institute of Oceanology,
Academy of Sciences, USSR

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